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ON THE SIMPLIFICATION OF THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH MORE THAN ONE TWO-PARAMETER INVARIANT SUBGROUP.

By S. D. Zeldin.

1. It is the purpose of this paper to extend the results obtained for the structure of groups with one two-parameter invariant subgroup* to groups having any number of two-parameter invariant subgroups.

Let $X_1, \cdots, X_r, X_{r+1}, \cdots, X_{r+2k}$ be the operators of the group G_{r+2k} whose order is $r + 2k$. Let this group have k invariant two-parameter subgroups which for simplicity will be taken to be represented by the operators

$$\begin{array}{lll} X_{r+1}, & X_{r+2} & (1), \\ X_{r+3}, & X_{r+4} & (2), \\ X_{r+5}, & X_{r+6} & (3), \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot \\ X_{r+2k-1}, & X_{r+2k} & (k). \end{array}$$

We then have

$$(X_i, X_j) = \sum_{s=1}^{r+2k} c_{ijs} X_s \qquad (i, j = 1, 2, \cdots, r),$$

$$(X_i, X_j) = \rho_{ij} \sum_{s=r+1}^{r+2} c_{ijs} X_s \quad (i = 1, 2, \cdots, r;\ j = r+1, r+2;\ \rho_{ij} \neq 0),$$

$$(X_i, X_j) = \rho_{ij} \sum_{s=r+3}^{r+4} c_{ijs} X_s \quad (i = 1, 2, \cdots, r;\ j = r+3, r+4;\ \rho_{ij} \neq 0),$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$(X_i, X_j) = \rho_{ij} \sum_{s=r+2k-1}^{r+2h} c_{ijs} X_s \ (i=1, 2, \cdots, r; j=r+2k-1, r+2k;\ \rho_{ij}\neq 0),$$

and

$$(X_i, X_j) = 0$$

$(i = r+1, \cdots, r+2k;\ j = i+1$, when $i = r +$ odd number, and $j = i - 1$, when $i = r +$ even number).

Denoting the operators of the adjoint of G_{r+2k} by the symbols $E_1, \cdots, E_r, E_{r+1}, \cdots, E_{r+2k}$, we have for the alternant of any two of these operators the same structural constants as for the alternant of the corresponding operators of the group G_{r+2k}.

We shall assume in what follows that the group G_{r+2k} is meroëdrically isomorphic with a simple group G_r of order r having one invariant spread (not flat).

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* See the preceding paper.

Denoting the operators of G_r by $Y_1, \cdots, Y_r$ and the operators of the adjoint of G_r by $A_1, \cdots, A_r$, we have

$$(Y_i, Y_j) = \sum_{s=1}^{r} c_{ijs} Y_s$$

and $\qquad (i, j = 1, 2, \cdots, r)$

$$(A_i, A_j) = \sum_{s=1}^{r} c_{ijs} A_s.$$

As I have already shown in my previous articles* the matrix $\Sigma\alpha_j A_j$, where, the α's being the parameters of the group G_r,

$$\Sigma\alpha_j A_j = \begin{vmatrix} \Sigma\alpha_j c_{j11}, & \cdots, & \Sigma\alpha_j c_{j1r} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \Sigma\alpha_j c_{jr1}, & \cdots, & \Sigma\alpha_j c_{jrr} \end{vmatrix},$$

is of nullity one. Let us now consider the equation

$$\left(\sum_{1}^{r+2k} \alpha_i X_i, \sum_{1}^{r+2k} \beta_j X_j\right) = \sum_{i=1}^{r+2k} \sum_{j=1}^{r+2k} \sum_{s=r+1}^{r+2k} \rho_{ij}\alpha_i\beta_j c_{ijs} X_s. \tag{A}$$

For the α's assigned, and ρ_{ij} and c_{ijs} $(i = 1, \cdots, r;\ s, j = r + 1, \cdots, r + 2k)$ easily calculated, it is clear that equation (A) has the following $2k + 1$ independent sets of β's for solutions:

$$\begin{array}{ll}
(1) & \beta_1 = \alpha_1, \quad \cdots, \quad \beta_r = \alpha_r, \quad \beta_{r+1} = \cdots = \beta_{r+2k} = 0 \\
(2) & \beta_1 = \cdots = \beta_r = 0, \quad \beta_{r+1} = 1, \quad \beta_{r+2} = \cdots = \beta_{r+2k} = 0 \\
(3) & \beta_1 = \cdots = \beta_r = \beta_{r+1} = 0, \quad \beta_{r+2} = 1, \quad \beta_{r+3} = \cdots = \beta_{r+2k} = 0 \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
(2k + 1) & \beta_1 = \cdots = \beta_{r+2k-1} = 0, \quad \beta_{r+2k} = 1.
\end{array}$$

The matrix of the coefficients of the β's in the equations obtained from equation (A), namely,

$$\begin{vmatrix}
\Sigma\alpha_i c_{i11}, & \Sigma\alpha_i c_{i21}, & \cdots, & \Sigma\alpha_i c_{ir1}, & \cdots, & \Sigma\alpha_i c_{i,\,r+2k,\,1} \\
\Sigma\alpha_i c_{i12}, & \Sigma\alpha_i c_{i22}, & \cdots, & \Sigma\alpha_i c_{ir2}, & \cdots, & \Sigma\alpha_i c_{i,\,r+2k,\,2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\Sigma\alpha_i c_{i1r}, & \Sigma\alpha_i c_{i2r}, & \cdots, & \Sigma\alpha_i c_{irr}, & \cdots, & \Sigma\alpha_i c_{i,\,r+2k,\,r} \\
\multicolumn{6}{l}{\Sigma\alpha_i c_{i,\,1,\,r+1}(1-\rho_{i1}), \quad \cdots, \quad \Sigma\alpha_i c_{i,\,r,\,r+1}(1-\rho_{ir}), \quad \cdots, \quad \Sigma\alpha_i c_{i,\,r+2k,\,r+1}(1-\rho_{i,\,r+2k})} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\multicolumn{6}{l}{\Sigma\alpha_i c_{i,\,1,\,r+2k}(1-\rho_{i1}), \quad \cdots, \quad \Sigma\alpha_i c_{i,\,r,\,r+2k}(1-\rho_{ir}), \quad \cdots, \quad \Sigma\alpha_i c_{i,\,r+2k,\,r+2k}(1-\rho_{i,\,r+2k})}
\end{vmatrix},$$

will *have for its nullity a number not less than* $2k + 1$. It follows, therefore, since every minor of the determinant $|\Sigma\alpha_i A_i|$ is also a minor of the determinant of the matrix obtained from equation (A), that the nullity of the matrix $\Sigma\alpha_i E_i$ is exactly equal to $2k+1$. The nullity of the matrix $\Sigma\alpha_i E_i$ is equal to the number of independent invariants of the adjoint of G. The following *theorem* can therefore be stated:

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* Loc. cit.

If the adjoint of G_r which is meroëdrically isomorphic with G_{r+2k} has one invariant, then the adjoint of G_{r+2k} has $2k+1$ independent invariants, one of which is the invariant of the adjoint of G_r.

2. We shall denote the $2k+1$ invariants by $V(\alpha_1, \cdots, \alpha_r)$, $W_1(\alpha_1, \cdots, \alpha_r, \cdots, \alpha_{r+2k})$, $\cdots$, $W_{2k}(\alpha_1, \cdots, \alpha_r, \cdots, \alpha_{r+2k})$; the equations

$$\begin{aligned} V(\alpha) &= 0 \\ W_1(\alpha) &= 0 \\ &\cdot\quad\cdot\quad\cdot\quad\cdot \\ &\cdot\quad\cdot\quad\cdot\quad\cdot \\ W_{2k}(\alpha) &= 0 \end{aligned}$$

will then represent $2k+1$ invariant spreads in the space of the adjoint of G_{r+2k}. Now, since the group G_{r+2k} has k invariant two-parameter subgroups, the adjoint of G_{r+2k} will leave invariant k straight lines, each one representing the corresponding invariant subgroup. We shall denote those lines, as in the previous paper, by the symbols $X_i \leftarrow\rightarrow X_j$ ($i = r+1, \cdots, r+2k$; $j = i+1$, when $i = r +$ odd number, and $j = i-1$, when $i = r +$ even number).

If the equations $W_1(\alpha) = 0, \cdots, W_{2k}(\alpha) = 0$ represent $2k$ invariant flats in the space of the adjoint of G_{r+2k}, then their common intersection (if there is any) will be an $(r-1)$-flat also invariant to the adjoint of G_{r+2k}. If that flat does not pass through any of the lines $X_i \leftarrow\rightarrow X_j$, then by Lie's theorem* we can take $\overline{X}_1, \cdots, \overline{X}_r, \overline{X}_{r+1} = X_{r+1}, \cdots, \overline{X}_{r+2k} = X_{r+2k}$, such linear functions of $X_1, \cdots, X_{r+2k}$ that the first $r-1$ operators form an invariant subgroup of order $r-1$, while the last $2k$ operators still generate the invariant two-parameter subgroups which we started with, i.e.,

$$\begin{aligned}
(\overline{X}_i, \overline{X}_j) &= \sum_{s=1}^{r-1}\bar{c}_{ijs}\overline{X}_s && (i = 1, \cdots, r-1;\ j = 1, \cdots, r), \\
(\overline{X}_i, \overline{X}_j) &= 0 && (i = 1, \cdots, r-1;\ j = r+1, \cdots, r+2k), \\
(\overline{X}_i, \overline{X}_j) &= \bar{\rho}_{ij}\sum_{r-1}^{r+2}\bar{c}_{ijs}\overline{X}_s && (i = r, r+1, r+2;\ j = r+1, r+2), \\
(\overline{X}_i, \overline{X}_j) &= \bar{\rho}_{ij}\sum_{r+3}^{r+4}\bar{c}_{ijs}\overline{X}_s && (i = r, r+3, r+4;\ j = r+3, r+4), \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
(\overline{X}_i, \overline{X}_j) &= \bar{\rho}_{ij}\sum_{r+2k-1}^{r+2k}\bar{c}_{ijs}\overline{X}_s && (i=r,\ r+2k-1,\ r+2k;\ j=r+2k-1,\ r+2k), \\
(\overline{X}_i, \overline{X}_j) &= 0 && (i = r+1, \cdots, r+2k;\ j = i \pm 2;\ j \neq r).
\end{aligned}$$

* Lie-Scheffers, Continuierliche Gruppen, p. 479.

If however not all flats have a common intersection, then the flat representing the common intersection of a few will enable us to form an invariant subgroup of G_{r+2k}.

Suppose now that $W_1(\alpha) = 0, \cdots, W_{2k}(\alpha) = 0$ are spreads of degrees $m_1, m_2, \cdots, m_{2k}$ respectively. Consider then the polar flats, with respect to each of the spreads

$$W_1(\alpha) = 0, \qquad \cdots, \qquad W_{2k}(\alpha) = 0,$$

of each of the lines $X_{r+1} \leftarrow\rightarrow X_{r+2}, \cdots, X_{r+2k-1} \leftarrow\rightarrow X_{r+2k}$. Taking, for instance, the polar flats of the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, with respect to

$$W_1(\alpha) = 0, \qquad \cdots, \qquad W_{2k}(\alpha) = 0,$$

we obtain $2k$ invariant r-flats in the space of the adjoint of G_{r+2k}. The same holds for all the other lines. Thus, there will be $2k^2$ invariant flats in addition to the k invariant flats formed by taking the polar flats of those lines with respect to the spread $V(\alpha) = 0$ ($V(\alpha)$ is the common invariant of the adjoints of G_r and G_{r+2k}).

If the common intersection of those $2k^2 + k$ invariant r-flats, if there is any, is an $(r - 1)$-flat, we can choose the operators of G_{r+2k} in such a way that $r - 1$ of them form an invariant subgroup of order $r - 1$. Or, more generally, if the common intersection is an $(r - i)$-flat $(1 \leqq i \leqq r - 1)$, the group G_{r+2k} will *have an invariant subgroup of order* $r - i$ by properly choosing the operators.

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